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Relative Stability for Control Systems with Adjustable Parameters

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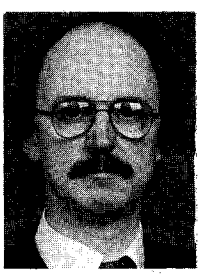
This paper presents an approach for finding regions of specified gain margins (GM) and phase margins (PM) in the parameter space of a single-input single-output (SISO) control system with adjustable parameters. The method uses the Nyquist stability criterion of encirclement of a point in the polar plot. The necessary boundaries of the specified GM or PM regions in the parameter space are fully treated by mapping a point related to specified GM or PM in the polar plane into the parameter space. A general problem with previous mapping techniques is identified where points in the parameter space do not necessarily satisfy the required GM or PM. A search-based technique is proposed to address this problem. Three examples illustrate the theoretical development.

I. Introduction

GAIN margin (GM) and phase margin (PM) are perhaps the two most important and widespread accepted criteria in analysis and design of practical control systems. These criteria are related to the open-loop transfer function of systems. In the robust stabilization problem of control systems, however, parameter space methods generally consider the closed-loop characteristic equation of the system.[1-3] Parameter space techniques were initially studied by Vishnegradesky for a third-order system using a graphical technique that was popularized by Nimark and was referred to as D-partition or D-decomposition[4-6] in its early development. This method actually establishes a direct correlation between the variable parameters of the closed-loop characteristic equation and the various important stability regions, e.g., open left half of the complex plane (OLHP). The method is directly suitable for graphical representation, in which case it is convenient to consider two parameters at a time.

The idea of correlating the free parameters of the open-loop transfer function of a system and the relative stability regions appears to have been initially given in Ref. 7. Unfortunately, the method was not developed in detail, and the idea of isolating the GM region in the parameter plane was not correct. A recent improvement to the idea was presented by introducing the gain-phase margin tester into the system,[8] which attempted to isolate specified GM and PM regions in the parameter space to allow a choice of the adjustable parameters. However, it is argued here that in this latter approach all the possible boundaries of the relative stability regions are not in fact considered, and some and possibly all points in the relative stability region do not necessarily satisfy the actual specified relative stability margin. In particular, the so called "singular boundaries" of the D-partition method[6] need to be considered. Finally, boundaries arising through frequency loci with re-entrant or convoluted characteristics also need to be considered with care. In this paper, a method for accounting for the



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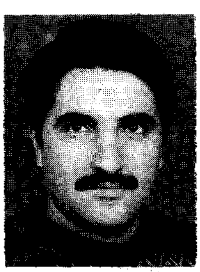

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singular boundaries of the D-partition method is established. In addition, a proposed search-based method handles systems with re-entrant or convoluted characteristics. The proposed techniques thus overcome the problems encountered in previous works and provide sufficient conditions for relative stability. For reasons of graphical output, however, the implementation of the method is largely limited to the investigation of the relative stability of a system through the variation of one, two, or three parameters at a time. The method proposed here is referred to as the open-loop D-partition (OLDP) method.

The proposed parameter space technique is distinguished from both existing conservative[9] and nonconservative[10] norm bound techniques. Existing norm bound methods are concerned with GMs or PMs for individual loops in multiple-input multiple-output (MIMO) systems and essentially result in identifying only conservative stability regions, albeit possibly least conservative boundaries for norm bound criterion. Furthermore, they relate margin boundaries to envelopes of general parametric uncertainty. The proposed parameter space approach, on the other hand, is concerned with the effect of specific variable parameters on GMs and PMs of SISO systems and results in the presentation of exact stability regions in the parameter space.

II. Nyquist Encirclement for Relative Stability Analysis

Consider the open-loop rational transfer function of the system shown in Fig. 1

$$L(s) = H(s)G(s) = \frac{L_N(s)}{L_D(s)} = \frac{a_l s^l + a_{l-1}s^{l-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \tag{1}$$

where $l \leq n$ and the coefficients a_i for $i = 0, 1, 2, \ldots, l$ and b_j for $j = 0, 1, 2, \ldots, n$ are real valued, linear affine functions of the parameter set $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$. It has been proposed[8] that the open-loop transfer function may be multiplied by a gain-phase margin tester $Ae^{-j\phi}$ to form a new open-loop transfer function. The characteristic equation of the newly formed system was then to be used to obtain the relative stability domain of the actual system. A new rigorous approach based on the mapping from points in the polar plane to the parameter space shows that this former method ignores certain crucial possible stability boundaries of the feedback system.

To explain this new approach, initially assume that the Nyquist plot of the system does not cross the negative real axis at multiple points in a re-entrant or convoluted form (Fig. 2a). Also assume that the portion of the unit circle in the third quarter of the polar plane is not crossed multiple times by the Nyquist plot in a re-entrant or convoluted form locus as shown in Fig. 2b (see Sec. IV). The characteristic polynomial of a feedback system is obtained from the numerator of $L(s) + 1$, where

$$L(s) + 1 = 0 \tag{2}$$

By putting $s = jw$ $(-\infty < w < +\infty)$ in Eq. (2), the D-partition boundaries can be constructed in the m-dimensional space,

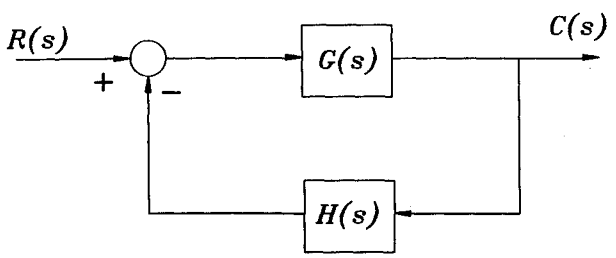

Fig. 1 Block diagram of a general feedback system.

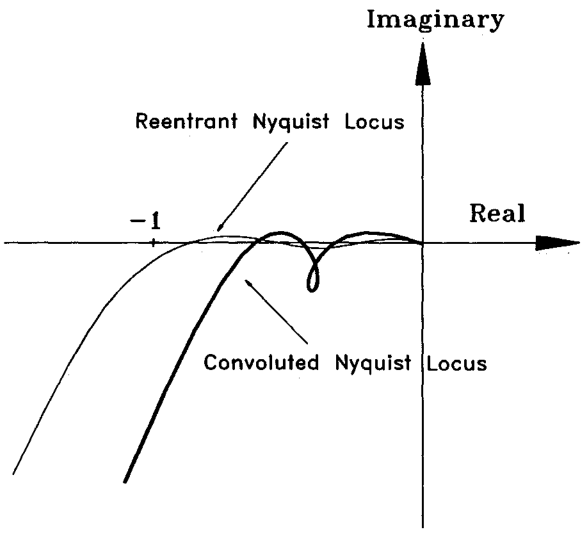

Fig. 2a Multiple crossing of a negative real axis.

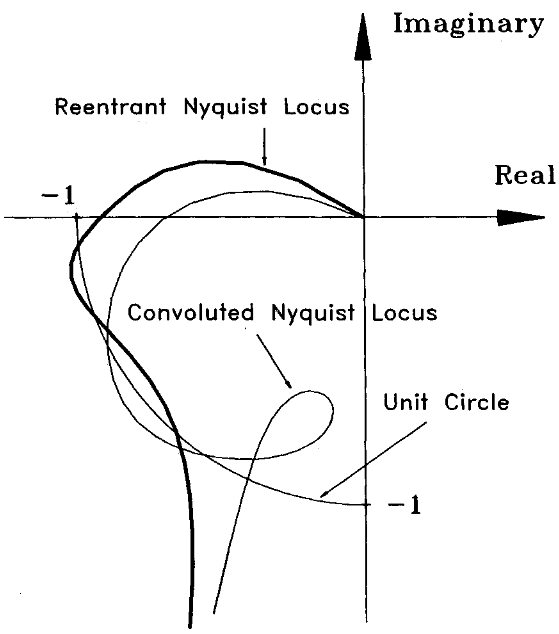

Fig. 2b Multiple crossing of a unit circle in the third quadrant.

and the domain of asymptotic stability (denoted D) may be specified, using the appropriate shading technique that relates the hatched side to the left half-plane or stable poles.[6,11]

To extend this idea, note that Eq. (2) can be written

$$L(jw) = -1 \tag{3}$$

This form emphasizes that the D-partition boundaries and the domain of asymptotic stability are determined by equating the open-loop transfer function of the system to negative unity as w increases from $-\infty$ to $+\infty$. In other words, the D-partition boundaries in the parameter space can be constructed by mapping the point $(-1, 0)$ in a polar plane onto the parameter space when w increases from $-\infty$ to $+\infty$. These boundaries divide the m-dimensional parameter space into two main regions. Any point in the first region corresponds to a set of parameters, which makes the Nyquist plot encircle the critical point the required number of times for asymptotic stability. The second region corresponds to those sets of parameters that make the Nyquist plot encircle the critical point a different number of times. The former, obviously, is generally the desired region. The stability domains determined by this map-

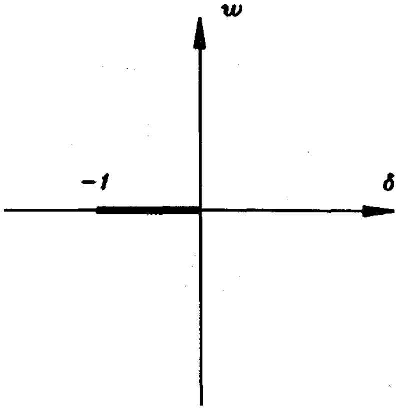

Fig. 3a Gain margin region.

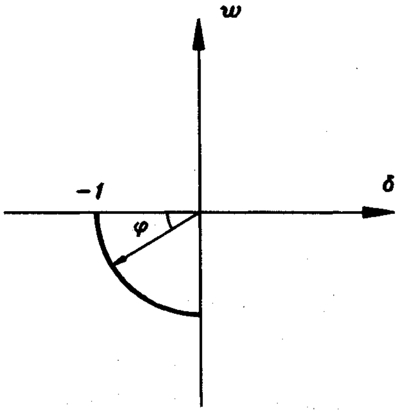

Fig. 3b Phase margin region.

ping and the shading technique are the domains of asymptotic stability.

Now it is possible to construct the boundaries of a particular relative stability margin and consequently determine the domains of relative stability, if any exist, by use of the more general form of Eq. (3)

$$L(jw) = a + jb \tag{4}$$

where $a + jb$ is an arbitrary point in the polar plane. By increasing w from $-\infty$ to $+\infty$, Eq. (4) may be used to define a locus that traces out, in the parameter space, a map of the point $a + jb$ in the polar plane. This transformation, using the same technique as the method of D-partition, isolates a region $\bar{D}$ in the parameter space for which the open-loop plots encircle the specified point $a + jb$ in the polar plane the required number of times.

Two sets of values of a and b are of particular interest.

1) The first set, which is shown by bold line in Fig. 3a, is the line of all points satisfying $b = 0$ and $a \in [-1, 0)$.

If $a = -1$, the mapping specifies all ξ in the parameter space that make the system marginally stable (the ordinary D-partition method). With a between -1 and some specified value less than zero, the mapping specifies the set of all ξ that make the open-loop transfer function have at least a specified GM of $K_g = -20 \log|a|$ dB. This region D_g in the parameter space is clearly a subset of the original D-partition stability region D ($D_g \subset D$). Clearly, the bigger the GM, the smaller the region of interest in the parameter space.

2) The second set, given in Fig. 3b, is the parametric relation, $a = -\cos\phi$, $b = -\sin\phi$, where $\phi \in [0, 90)$. If $\phi = 0$, then the mapping specifies regions in the parameter space for which the system is marginally stable (the ordinary D-partition method). On increasing the angle ϕ, the mapping from the polar plane to the parameter space specifies the set of all ξ that make the open-loop transfer function have a PM of at least ϕ. Again, this region in the parameter space D_p is clearly a subset of the original D-partition stability region ($D_p \subset D$). A bigger PM also makes the corresponding region in the parameter space smaller.

To achieve both the specified GM and PM, the intersection of the two desired regions D_g and D_p in the parameter space D_{gp} gives the set of all ξ needed for analysis and design. In other words, D_g and D_p should be considered separately and then the intersection formed

$$D_{gp} = D_g \cap D_p \tag{5}$$

Since a and b can be determined directly by specific GM or PM requirements, the adjustable parameters can be tuned inside the D_{gp} so that the system satisfies a specified measure of relative stability margin.

III. GM and PM Boundaries in the Parameter Space

The open-loop D-partition analysis is, as noted in the previous section, based on a mapping procedure that transforms a point from the polar plane onto the parameter space. Then, using a similar technique to that of the method of D-partition, the relative stability region is specified. If in the general case the open-loop transfer function has the form of Eq. (1), the implicit mapping function can be obtained from

$$L(s) = L_N(s)/L_D(s) = (a + jb) \tag{6}$$

For convenience the function $H(s)$ is defined and equated to zero as

$$H(s) \equiv L_N(s) - (a + jb)L_D(s) = 0 \tag{7}$$

which may conveniently be called the "relative characteristic equation" (RCE). The function has the polynomial form of

$$H(s) = A_n s^n + A_{n-1}s^{n-1} + \cdots + A_1 s + A_0 = 0 \tag{8}$$

$$A_i \neq 0$$

where $s = jw + \delta$ is the Laplacian complex variable, and where the complex coefficients A_i for $i = 0, 1, 2, \ldots, n$ have real and imaginary parts that are linear affine functions of m real parameters $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ given by

$$A_i = c_{i0} + jd_{i0} + (c_{i1} + jd_{i1})\xi_1 + \cdots + (c_{im} + jd_{im})\xi_m \tag{9}$$

where c_{il} and d_{il} for $l = 0, 1, 2, \ldots, m$ are real scalar values. The assumption of linearity implies that there are no cross product terms. This covers a wide set of practical situations. Obvious extensions by numerical methods nevertheless exist for many cases where there is such cross coupling.

To a given point in the polar plane $a + jb$ subject to the previous constraints, there may correspond a domain in the parameter space so that for any ξ in that domain a specified relative stability is satisfied. Clearly, the domain of relative stability in each of the two cases (specified GM and specified PM) is determined by the domain of asymptotic stability of the RCE of Eq. (8) evaluated for the appropriate values of a and b. At the boundaries of this domain the mapping equation $H(s) = 0$ is satisfied. Thus, for any point ξ in the hull of the domain, the polar plot of $L(s)$ will pass through the point $a + jb$.

Substituting the Laplace variable s with jw in Eq. (8) yields

$$H(jw) = A_n(jw)^n + A_{n-1}(jw)^{n-1} + \cdots + A_1(jw) + A_0 = 0 \tag{10}$$

For $-\infty \leq w \leq +\infty$ the boundaries of the corresponding relative stability domain may be determined by the following surfaces in the parameter space.

Surface 1. An $(m-2)$-dimensional hyperplane, $\Delta_{A0}=0$, which corresponds to an $H(s)$ with a real zero on the origin of the s plane, so that since $H(0)=0$

$$\Delta_{A0}=A_0=0 \tag{11}$$

When A_0 has only a real part or only an imaginary part, the dimension of the hyperplane is $(m-1)$.

Surface 2. An $(m-2)$-dimensional hyperplane, $\Delta_{An}=0$, which corresponds to an $H(s)$ with a real zero at infinity in the s plane, so that since $H(\pm\infty)=0$

$$\Delta_{An}=A_n=0 \tag{12}$$

When A_n contains only a real part or only an imaginary part, the dimension of the hyperplane $\Delta_{An}=0$ is $(m-1)$.

Surface 3. An $(m-1)$-dimensional hyperplane, $\Delta_{+w}=0$, which is generated by the movement of an $(m-2)$-dimensional hyperplane and corresponds to $H(s)$ with a pure imaginary zero when $w\in(0,+\infty)$ given by

$$s=jw_1 \qquad w_1>0 \tag{13}$$

so that

$$H(s)=(s-jw_1)V_1(s) \tag{14}$$

where $V_1(s)$ is an $(n-1)$-order function.

Surface 4. An $(m-1)$-dimensional hyperplane, $\Delta_{-w}=0$, which is generated by the movement of an $(m-2)$-dimensional hyperplane and corresponds to $H(s)$ with a pure imaginary zero when $w\in(0,-\infty)$ given by

$$s=jw_2 \qquad w_2<0 \tag{15}$$

so that

$$H(s)=(s-jw_2)V_2(s) \tag{16}$$

where $V_2(s)$ is an $(n-1)$-order function.

The two last hyperplanes are clearly not the same in a polynomial with complex coefficients, since the complex roots are not necessarily in conjugate pairs. The corresponding surfaces can then be determined by applying the condition that the summations of the reals and the imaginaries of Eq. (10) must go to zero independently. For example, when there are only two parameters ξ_1 and ξ_2, Eq. (10) can be rewritten as the following simultaneous equations

$$R(jw)=P_R\xi_1+Q_R\xi_2+R_R=0 \tag{17}$$

$$I(jw)=P_I\xi_1+Q_I\xi_2+R_I=0 \tag{18}$$

where $R(jw)$ and $I(jw)$ are the real and the imaginary parts of $H(jw)$.

Equations (17) and (18), which comprise two equations in the two unknown parameters $\xi_1(\boldsymbol{a},\boldsymbol{b},w)$ and $\xi_2(\boldsymbol{a},\boldsymbol{b},w)$, may be solved for these parameters provided that the determinant given by

$$\Delta(w)=\begin{vmatrix} P_R & Q_R \\ P_I & Q_I \end{vmatrix}=P_RQ_I-P_IQ_R \tag{19}$$

does not vanish for $-\infty<w<0$ and $0<w<+\infty$ except at some finite number of values of w.

Surface 5. For those finite number of values of w for which the determinant vanishes the corresponding surfaces, denoted Δ_s, may be found by substituting the appropriate values of w into Eqs. (17) or (18).

The surfaces corresponding to Δ_{A0}, Δ_{An} and Δ_s are the singular surfaces, whereas the surfaces corresponding to Δ_{+w} and Δ_{-w} are the nonsingular surfaces. Note that none of these singular surfaces are identified in previous techniques.[7,8]

To specify the domain of desired relative stability, the procedure of shading (hatching) is the same as that in the ordinary D-partition method.[5,6] The shaded side corresponds to the stable side of the imaginary axis in the s plane. The region in which all the shadings are on the inside of the hyperplane is the region of stable roots of $H(s)$ and the domain of specified relative stability of $L(s)$. This region may in fact be determined by a variety of alternative methods.[12]

Example 1. The method and the importance of the singular boundaries are illustrated by determining all possible relative stability boundaries for the linear unitary feedback system with open-loop transfer function $L(s)$ given by

$$L(s)=\frac{3\eta-2}{\beta s^3+(2\beta+1)s^2+(\beta+1)s} \tag{20}$$

where β and η are free parameters of the system. The aim of the analysis is to determine the domain of relative stability in the parameter space to obtain a minimum GM of 10 dB and a minimum PM of 45 deg. The relative characteristic equation $H(s)$ of the system is

$$H(s)=(\boldsymbol{a}+j\boldsymbol{b})\left[\beta s^3+(2\beta+1)s^2+(\beta+1)s\right]-3\eta+2=0 \tag{21}$$

Substituting s with jw yields

$$H(jw)=(-j\boldsymbol{a}+\boldsymbol{b})\beta w^3-(\boldsymbol{a}+j\boldsymbol{b})(2\beta+1)w^2$$
$$+(j\boldsymbol{a}-\boldsymbol{b})(\beta+1)w-3\eta+2=0 \tag{22}$$

For a PM of 45 deg the corresponding stability boundaries are

1) $\Delta_{A0}=-3\eta+2=0 \Rightarrow \eta=0.667$.
2) $\Delta_{An}=(j0.707-0.707)\beta=0 \Rightarrow \beta=0$.
3) For Δ_{+w} and Δ_{-w} Eq. (22) implies that

$$R(jw)=(0.707w^2-1.414w-0.707)w\beta+3\eta$$
$$-0.707w^2-0.707w-2=0 \tag{23}$$

$$I(jw)=(-0.707w^2-1.414w+0.707)w\beta+0.707w^2$$
$$-0.707w=0 \tag{24}$$

The appropriate determinant corresponding to these simultaneous equations (which determine the nonsingular boundaries) is $\Delta(w)=2.1213w(w^2+2w-1)$. The determinant is zero if $w=0$, $w=0.414$, or $w=-2.414$. When $\Delta(w)\neq 0$ the D-partition boundary is given by the equations

$$\beta=\frac{1-w}{w^2+2w-1} \tag{25}$$

$$\eta=\frac{1.414w^4+3.414w^2+4w-2}{3(w^2+2w-1)} \tag{26}$$

In this case the coefficients of the RCE are complex, thus there are two different boundaries, one for $-\infty<w<0$ and one for $0<w<\infty$.

4) When $\Delta(w)=0$, the corresponding singular boundaries Δ_s are

$$\eta=0.667 \quad \text{when} \quad w=0 \tag{27}$$

$$\eta=\infty \quad \text{and} \quad \beta=\infty \quad \text{when} \quad w=-2.414 \tag{28}$$

$$\eta=\infty \quad \text{and} \quad \beta=\infty \quad \text{when} \quad w=0.414 \tag{29}$$

The different domains including the domains of relative stability in this case are shown in Fig. 4, where $D(k, 3-k)$ corresponds to the set of all values of parameters resulting in k roots of the RCE in the left half of the complex plane and $3-k$ roots in the right half of the complex plane.[5] Thus, $D(3,0)$ is the stability region of the RCE, and also the region of 45 deg PM of the system D_p. Note that without singular boundaries D_p cannot be specified.

For a GM of 10 dB the corresponding stability boundaries can similarly be found to be

1) $\Delta_{A0} = -3\eta + 2 = 0 \Rightarrow \eta = 0.667$.
2) $\Delta_{An} = 0.316\beta = 0 \Rightarrow \beta = 0$.
3) Δ_{+w} and Δ_{-w}

$$\eta = \frac{0.316w^4 + 2.316w^2 - 2}{3(w^2-1)} \tag{30}$$

$$\beta = \frac{1}{(w^2-1)} \tag{31}$$

These equations define the nonsingular boundaries. Since the coefficients of the RCE in this case are real ($b = 0$), the two boundaries for $-\infty < w < 0$ and $0 < w < \infty$ are the same. This fact is clear from Eqs. (30) and (31). (In both equations the power of w is even.)

4) When $\Delta(w) = 0.948w(w^2-1)$ is equal to zero, the singular lines Δ_s are

$$\eta = 0.667 \quad \text{when} \quad w = 0 \tag{32}$$

$$\beta = \pm\infty \quad \text{and} \quad \eta = \pm\infty \quad \text{when} \quad w = \pm 1 \tag{33}$$

The open-loop D-partition boundaries and the different domains for this case are shown in Fig. 5. The intersection of the domains of relative stability of the two different cases (D_{gp}) is shown in Fig. 6. This domain satisfies both required GM and PM. It is noted that this example actually results in two distinct stability ($D(3,0)$) regions.

IV. Re-Entrant and Convoluted Crossings

Additional complexities arise in the application of the OLDP method to systems within whose Nyquist plot some type of multiple crossings of the negative real axis or part of the unit circle in the third quarter of the polar plane occur. In particular, further considerations arise in the case of re-entrant or convoluted crossings. This type is where the locus crosses the negative real axis or unit circle more than once without encircling the origin between the crossing points. Conditionally stable systems exhibit this type of crossing of the negative real axis.[13,14] For those systems where the Nyquist plot does not cross the negative real axis at multiple points, and also does not exhibit multiple crossings of the unit circle within the third quarter of the polar plane, there is no additional difficulty. Where there are such crossing points, care must be taken because the isolated region in the parameter space then may not

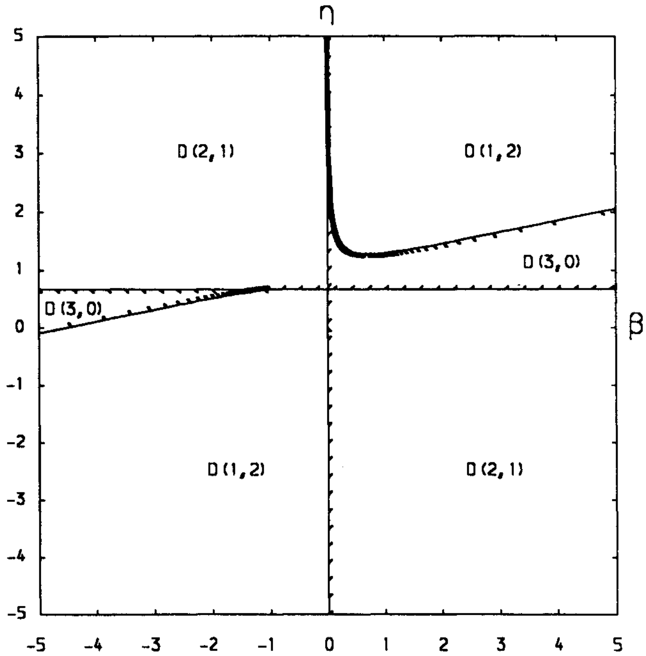

Fig. 5 OLDP diagram for 10 dB GM in the example in Sec. III. [$D(k, 3-k)$ as in text with $D(3, 0)$ the relative stability region].

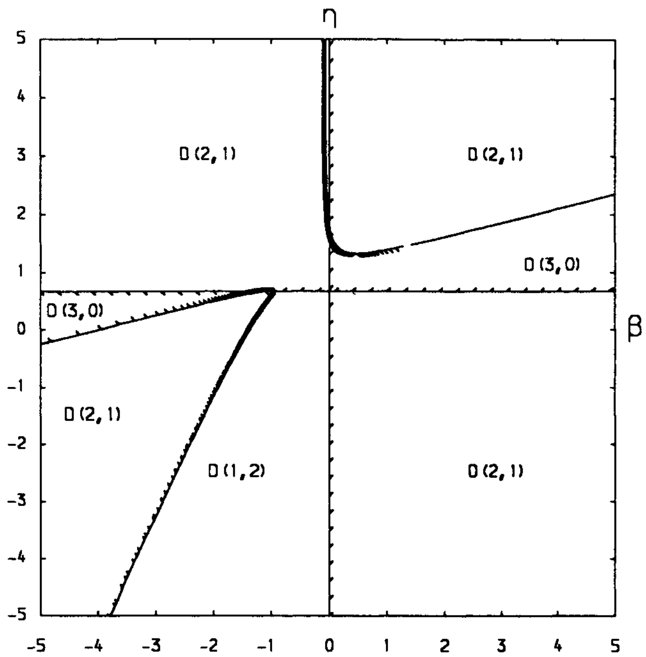

Fig. 4 OLDP diagram for 45 deg PM in the example in Sec. III. [$D(k, 3-k)$ as in text with $D(3, 0)$ the relative stability region].

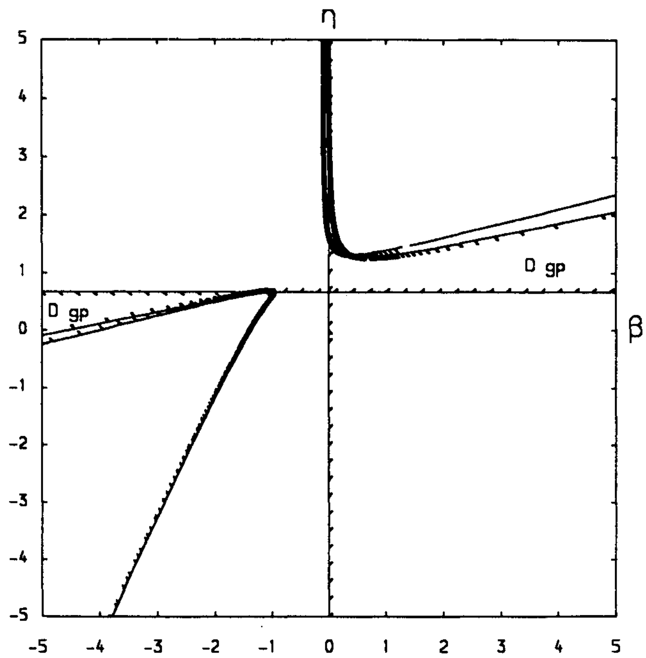

Fig. 6 OLDP diagram for 10 dB GM and 45 deg PM in the example in Sec. III. (with D_{gp} the combined desired stability region).

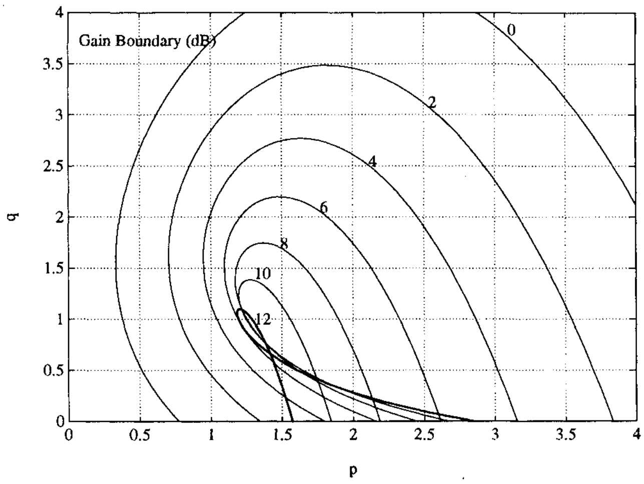

Fig. 7 OLDP diagram for PM boundaries form Nyquist locus with multiple crossing of the negative real axis.

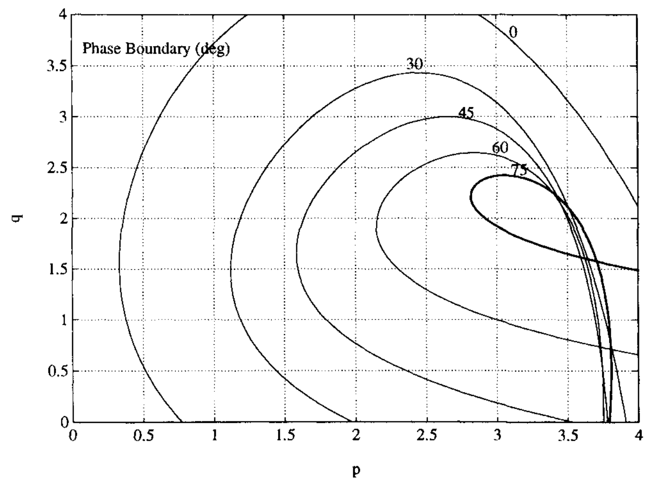

Fig. 8 OLDP diagram for GM boundaries from Nyquist locus with multiple crossings of the unit circle in the third quadrant.

represent the desired region. The Nyquist plot in the case of stable open-loop systems may cross the negative real axis to the left of the desired point at least two times. This situation means that, though the specified point in the polar plane is encircled the required number of times, there may be another portion of the negative real axis to the left of the point being mapped that is encircled a different number of times. If the critical point is included in this portion, then the system is unstable. If the portion is between the critical point and the desired point on the negative real axis then the mapped GM is not satisfied.

To safeguard against this problem, it is always possible to plot boundaries for several reduced GMs and PMs together with the asymptotic stability boundaries. If the regions for bigger GMs or PMs are completely contained within the smaller margin regions and all the different GM or PM regions are contained within the asymptotic stability region, re-entrant or convoluted crossings do not occur. However, if the bigger GM or PM region is not completely contained within the smaller one, then there exists such a multiple crossing for all those values of parameters contained in the higher GM or PM regions that are also not contained in the smaller GM or PM region.

In general, adjustable system parameters should be selected to be within the intersection of the appropriate GM or PM regions. The reason is that systems with the previously mentioned Nyquist crossings (including the conditionally stable systems) are generally not desirable.[14] A search method based on repeated plots of GM or PM for different levels of stability offers a useful way to avoid systems with such undesirable characteristics.

In multiple parameter cases where more than three parameters are to be selected, re-entrant crossings are less easily identified. On the other hand, the method is readily extended to systems with significant time delays despite the infinite number of crossings in such cases. This extension is possible because time delays do not change the form of the encirclement.

Example 2. As an example of a system with a re-entrant locus, consider the open-loop transfer function

$$L(s) = \frac{10(ps^2 + qs + 2.6)}{s(s+8)(s^2 + 1.14s + 4.5)(s^2 + 0.83s + 1)} \tag{34}$$

The GM boundaries of the system are plotted for different values of GM in Fig. 7. The 12 dB GM region in this case is not completely inside the 8 dB and 10 dB GM regions but it is completely inside the 6 dB GM region. Those points that are not within the smaller GM regions (we refer to the set of these points as the outside region) do not satisfy the desired GM. The reason why these points do not satisfy the desired GM is that the Nyquist plot of the system has several re-entrant crossings of the negative real axes. The choice of a large GM (e.g., 12 dB) means that there are two additional crossings between the critical point and the point related to the specified GM if the parameters are not chosen inside the intersection region. In this case the determined region includes points with worse GM.

A similar problem exists for PMs with re-entrant crossings on the unit circle (see Fig. 8). This situation can similarly be dealt with by determining the intersection of the appropriate PM regions.

Figure 9 shows the system's Nyquist plot for three different sets of parameter values for p and q. It can be seen from the figure that two different portions of the negative real axis are encircled by the Nyquist plot between the critical point and the origin if the parameters are chosen from the outside region of the OLDP diagram of Fig. 7. Also, two different portions of the unit circle in the third quadrant are encircled by the Nyquist plot if the parameters are chosen from the outside region of the OLDP diagram of Fig. 8.

Example 3. As a practical example consider a satellite control problem.[15,2] The satellite is modeled by two masses connected by a spring with torque constant k and viscous damping d. The corresponding transfer function of the system is

$$G(s) = \frac{s^2 + 10ds + 10k}{s^2(s^2 + 11ds + 11k)} \tag{35}$$

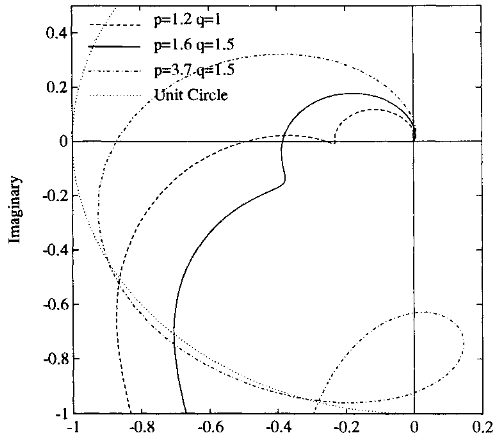

Fig. 9 Nyquist diagrams for different sets of parameters of example 1 in Sec. IV.

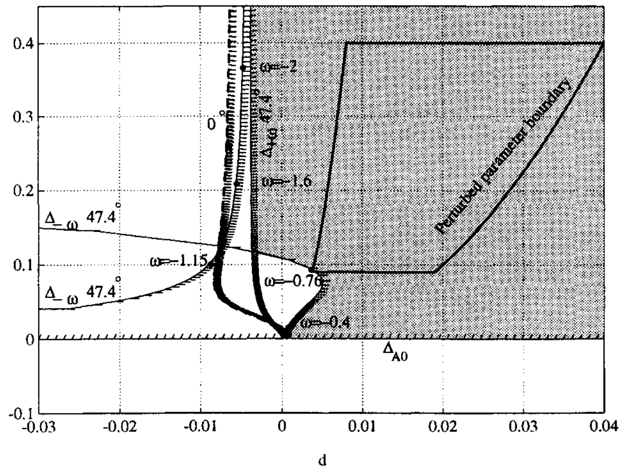

Fig. 10 OLDP diagram for 0 and 47.4 deg PM for the controlled satellite. (Relative stability region is shown as gray.)

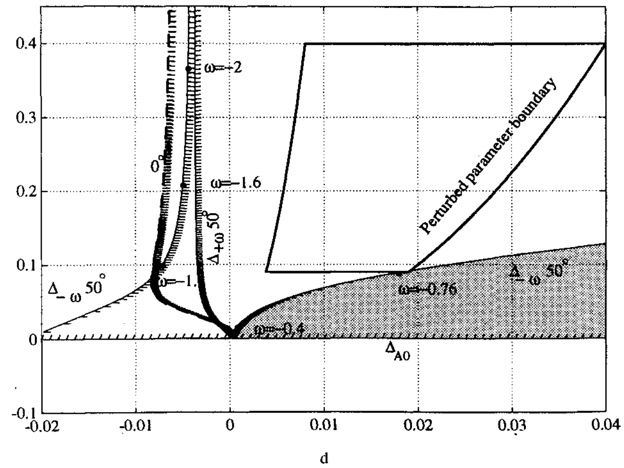

Fig. 11 OLDP diagaram for 0 and 50 deg PM for the controlled satellite. (Relative stability region is shown as gray.)

The problem is to design a controller to satisfy about 50 deg of PM having an open-loop gain crossover frequency of $w_c = 0.5 \text{ rad}\cdot\text{s}^{-1}$ for all parameter values in the region

$$0.09 \le k \le 0.4 \tag{36}$$

$$0.04\sqrt{\frac{k}{10}} \le d \le 0.2\sqrt{\frac{k}{10}} \tag{37}$$

A possible compensator is[15]

$$C(s) = 0.5(1.4s + 1) \tag{38}$$

The OLDP diagram of the system with compensator is plotted in Figs. 10 and 11 for 47.4 deg PM and 50 deg PM, respectively. In each case the perturbed parameter region is delineated by bold lines. Both figures also show the marginal stability boundary at 0 deg. Particular values of frequency are specified on the Δ_{-w} PM boundary in both figures. The OLDP diagrams indicate that the crossover frequency is well above $0.5 \text{ rad}\cdot\text{s}^{-1}$. In Figs. 10 and 11, Δ_{+w} for 47.4 deg and 50 deg are almost identical, however Δ_{-w} for these two phase margins are quite different. Also note that the line $k = 0$ is a singular line in each case. Figure 10 indicates that the system satisfies 47.4 deg PM for all parameter perturbations. Figure 11 indicates that a 50 deg PM, however, is not satisfied in any of the uncertainty bound region.

V. Conclusion

In this paper, an extension of the method of D-partition for finding GM and PM regions in the parameter space is developed. The extended method establishes a direct connection between the perturbed parameters of the open-loop transfer function in the parameter space and the minimum GM and PM criteria for the system. A method is proposed to adjust the free parameters of a system to avoid conditional stability. The proposed technique safeguards the system against the instability (or relative instability) to which many complex systems are prone.

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